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LA-UR -90-2888

SEP 07 1990

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LA-UR--90-2888

DE90 016438

SUBMITTED TO: **1990 LINAC Conference**
Accelerator Technology Division
MS H811
Los Alamos, NM 87544
Sept. 9-14, 1990 (Albuq. Hilton Hotel)

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HIGH ORDER OPTICS OF MULTIPOLE MAGNETS *

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ABSTRACT

We have developed a new capability to compute third and fifth order Lie algebraic transfer maps for a family of realistic multipole magnets, including dipoles. The general Hamiltonian is expanded symbolically to arbitrary order. The vector potential off axis, for a given multipole symmetry, is determined from the appropriate magnetic field gradients and their longitudinal derivatives on axis.

Subroutines to compute the required gradients are available for Halbach REC quadrupoles, and for general multipoles with the current distribution on a cylindrical surface specified by a shape function. This function can be supplied by the user, or selected from internal options.

Both the reference trajectory, and the map about it are calculated by numerical integration through the general magnetic field, using modular GENMAP software. This allows the calculation of curved reference trajectories in a general dipole magnet, as well as offset reference trajectories needed for misalignment tolerence studies. These new calculational capabilities have been added to the MARYLIE Lie Algebraic beam optics design code.

EQUATIONS OF MOTION

In magnetic optics, the particle coordinates commonly used are dimensionless deviations from a reference trajectory, defined as the path of a selected reference particle through the system. The longitudinal position s of the reference particle is taken as the independent variable in place of the time, and the deviations in time of flight ($\tau = c\Delta t$) and energy ($p_\tau = -\Delta E/p_0 c$) relative to the reference particle are taken as new longitudinal coordinates. The transverse momenta are scaled with respect to the design particle momentum: $P_x = p_x/p_0$, $P_y = p_y/p_0$. For convenience, we will often denote the vector of these new phase space coordinates by $\vec{z} \equiv (x, P_x, y, P_y, \tau, p_\tau)$.

When expressd in these new beam coordinates, the Hamiltonian turns out to be essentially the old p_z [1]:

$$H = -\frac{A_s}{B\rho} - \sqrt{1 + p_\tau^2 - 2\frac{p_\tau}{\beta_0} - (P_x - \frac{A_x}{B\rho})^2 - (P_y - \frac{A_y}{B\rho})^2} \tag{1}$$

The special role of the z-component of the vector potential should be noted. In regions where the transverse components of the vector potential vanish, the Hamiltonian has the form of a kinetic term plus a potential $A_z/B\rho$.

Hamiltons equations of motion in beam coordinates may be written:

$$\frac{dz_a}{ds} = -\{H, z_a\} \equiv -:H: z_a, \tag{2}$$

where the new symbol $:H:$ denotes the "Lie operator" associated with the Hamiltonian. The action of this differential operator on any function $f(\vec{z})$ is defined by the Poisson bracket with the Hamiltonian: $\{H, f\}$.

LIE ALGEBRAIC REPRESENTATION OF TRANSFER MAPS

The idea of a transfer map is simply that the final state of a particle is some function of its initial state: $\vec{z}_f = \vec{F}(\vec{z}_i)$. The transfer map can also be viewed as an operator $\mathrm{M}(s)$ that converts initial phase space coordinates (at $s = 0$) into final ones: $\vec{z}(s) = \mathrm{M}(s)\vec{z}(0)$. Hamilton's equations of motion for $\vec{z}(s)$ imply corresponding formal equations of motion for this operator, with the initial conditions $\mathrm{M}(0) = \mathrm{I}$, the identity. The equations have the form

$$\frac{d\mathrm{M}(s)}{ds} = -:H: \mathrm{M}(s). \tag{3}$$

For a beamline element of length L in which the Hamiltonian H is constant, this equation may be formally integrated to yield a map of the form

$$M = e^{-L:H:} \tag{4}$$

The exponential of a Lie operator is called a "Lie Transformation", which may be defined (for arbitrary $f(\vec{z})$) by the usual power series for an exponential. The application of this Lie transform to a general function $g(\vec{z})$ thus takes the form

$$e^{:f:} g = g + \{f, g\} + \frac{1}{2}\{f, \{f, g\}\} + \cdots \tag{5}$$

The GENMAP [2,3] routines in MARYLIE 3.0 represent the map of an optical system in the "reverse factorized" form

$$M = e^{:f_3:} e^{:f_4:} e^{:f_5:} \cdots, \tag{6}$$

where the $f_m(\vec{z})$ are homogeneous polynomials of order m. Substitution of this explicit form into the equation of motion of the map reduces it to a set of coupled nonlinear differential equations for the Lie polynomials. The Hamiltonian itself, which depends on the vector potential,

* Work supported and funded by the U.S. Department of Defense, Army Strategic Defense Command, under the auspices of the U.S. Department of Energy

must also be Taylor expanded into a series of homogeneous polynomials:

$$H(\vec{z},s) = \sum_{n=1}^{\infty} H_n(\vec{z},s). \tag{7}$$

Finally, the first order behavior of the system is taken out by transforming to the "interaction representation" in which only deviations from first order motion are considered. The Interaction Representation of the n^{th} order part of the Hamiltonian is:

$$H_n^I(z,s) \equiv H_n(R(s)\,z,s), \tag{8}$$

where $R(s)$ is the linear (matrix) part of the map $\mathbf{M}(s)$, whose motion is determined by $H_2(s)$ alone.

After all of this, the equations for f_3 and f_4 are:

$$\frac{\partial f_3(z,s)}{\partial s} = -H_3^I(z,s), \tag{9}$$

$$\frac{\partial f_4(z,s)}{\partial s} = -H_4^I(z,s) + \frac{1}{2}\left[f_3(z,s), \frac{\partial f_3(z,s)}{\partial s}\right]. \tag{10}$$

These equations are further separated power by power into coupled non-linear differential equations for the 209 polynomial coefficients (for m = 2, 3, and 4) that represent the transfer map to third order, as used by MARYLIE 3.0. The GENMAP routines numerically integrate these 209 differential equations.

The fifth order code MARYLIE 5.0 uses m = 2 through 6, for a total of 923 equations to be integrated. The additional equations of motion for the f_m are more complicated, involving multiple Poisson brackets.

REPRESENTATION OF THE VECTOR POTENTIAL

Given the Fourier expansion of the scalar potential

$$V(r,\theta,z) = \sum_{m=1}^{\infty} V_m(r,z)\sin(m\theta), \tag{11}$$

a vector potential giving the same field is

$$\begin{aligned} A_z &= -\sum_{m=1}^{\infty} \frac{\cos(m\theta)}{m} r \frac{\partial}{\partial r} V_m(r,z) \\ A_r &= \sum_{m=1}^{\infty} \frac{\cos(m\theta)}{m} r \frac{\partial}{\partial z} V_m(r,z). \end{aligned} \tag{12}$$

Here we have choosen a gauge where $A_\theta = 0$. The scalar potential off-axis may be written

$$V_m(r,z) = r^m \sum_{l=1}^{\infty} \frac{(-1)^l (m-1)!}{l!(l+m)!}\left(\frac{r}{2}\right)^{2l}\left(\frac{\partial}{\partial z}\right)^{2l} g_m(z), \tag{13}$$

where

$$g_m(z) = \lim_{r\to 0} \frac{m! V_m(r,z)}{r^m} \tag{14}$$

represents the profile of the mth multipole.

This is a general solution to Maxwell equations order by order, for arbitrary $g_m(z)$. The problem is thus reduced to computing the generalized field gradients on axis for realistic magnet models. We report here on the implementation of a family of magnet models representable by current sheets on a cylindrical surface.

CURRENT SHEET MAGNETS

Cylindrical current sheets can be used not only to represent radially thin windings on a cylinder, but also to calculate arbitrary fields in the source-free volume bounded by a cylindrical surface. In the latter case, fictitious surface currents that produce the same interior fields as complicated outside sources replace them for rapid field calculation. (For some simple volume field sources, it can be better to work directly with analytical expressions for the field from the source). Continuous currents on the surface of a cylinder of radius a can be represented by a stream function $\Psi(\phi,z)$ of the surface coordinates as follows [4]:

$$j_\phi = -NI\frac{\partial \Psi(\phi,z)}{\partial z}, \quad j_z = \frac{NI}{a}\frac{\partial \Psi(\phi,z)}{\partial \phi}. \tag{15}$$

The quantity NI is introduced to make Ψ dimensionless. The above prescription produces currents that are automatically divergence-free. The stream function can be used to find a set of discrete turns that approximate a continuous distribution, since current streamlines are contours of constant Ψ; to make spiral windings, successive closed turns are cut and transitions between them are inserted. The stream function can be written as a sum of Fourier components as follows:

$$\Psi(\phi,z) = \sum_{m=1}^{\infty} w_m(z)\sin(m\phi). \tag{16}$$

The $w_m(z)$ are called shape functions. The $m = 0$ case is excluded here because it requires special treatment. If only a single m value is present in Eq.16 and the boundaries are rotationally symmetric, the resultant field has m-pole θ dependence everywhere. For open boundaries, the scalar potential produced by the currents of Eq.15 is

$$V(r,\theta,z) = \frac{\mu_0 NIa}{\pi}\sum_{m=1}^{\infty}\sin(m\theta)\int_{-\infty}^{+\infty} w_m(x)G_m(r,z,x)dx. \tag{17}$$

The Green's function $G_m(r,z,x)$ has the form

$$G_m(r,z,x) = \frac{\partial}{\partial a}\left[\frac{1}{2\sqrt{ar}}Q_{m-\frac{1}{2}}\left(\frac{a^2+r^2+(z-x)^2}{2ar}\right)\right], \tag{18}$$

where $Q_{m-\frac{1}{2}}$ is a Legendre function of the second kind of half-integral order. (For the closed-boundary case of a surrounding cylinder of infinitely permeable material, hybrid series solutions containing both ordinary and hyperbolic

Bessel functions have been obtained). Expressions for the magnetic field and vector potential are easily derived from Eq.17. For calculation of fields, etc., the $Q_{m-\frac{1}{2}}$ are not directly used; instead, the $\frac{1}{2\sqrt{ar}}$ factor cancels a factor in the hypergeometric series expression for $Q_{m-\frac{1}{2}}$; the resultant expression is regular as $r \to 0$. The generalized on-axis gradient for a single multipole m was defined in Eq.14. This limit of Eq.17 is

$$g_m(z) = \frac{\mu_0 N I a (2m-1)!!}{(m-1)! 2^{m+1}} \int_{-\infty}^{+\infty} w_m(x) K_m(z,x) dx \quad (19)$$

where

$$K_m = \frac{\partial}{\partial a} \left\{ \frac{a^m}{[(z-x)^2 + a^2]^{m+1/2}} \right\}. \quad (20)$$

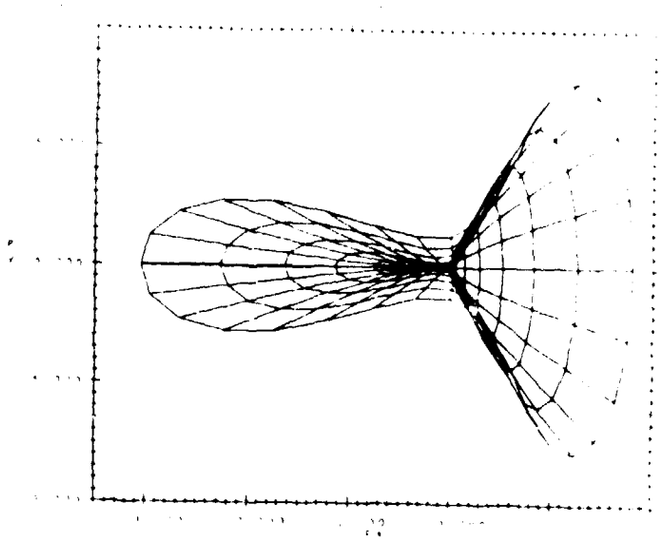
If $w_m(x)$ is a piecewise continuous polynomial, exact analytic integration of Eq.19 can be done. An exact quadrature subroutine for polynomials of up to the 4th degree in Eq.19 is used to calculate g_m and its z derivatives to arbitrary order. This routine is called by routines for shape functions of several types, including flattops with square or rounded edges, a symmetric quartic, etc. Lambertson coils are represented by two shape functions, the first for the fundamental with a given m value for the angular dependence, and second for the first allowable harmonic, with an angular dependence of 3m. The two shape functions are represented by a series of parabolic arcs precomputed in an initialization call. Finally, a user may provide a shape function in the form of a set of points; these points are then interpolated by parabolic arcs. The above routines have been implemented in MARYLIE, along with a routine for Halbach REC quadrupoles. The implementation is quite flexible, allowing almost arbitrary sequences of overlapping elements, as well as fitting and optimization.

EXAMPLE

As an illustration of the new capabilities, we show some calculations on a point-to-parallel telescope consisting of a quadrupole doublet with a concentric bending dipole. After setting the quadrupoles to obtain a focal point 3.0 m upstream with the dipole off, we turn on the bending dipole, and refocus the quadrupoles to compensate for the fringe fields of the bend. This excercises the curved reference trajectory, overlapping fringe fields, and fitting capabilities of the new code. This is the description of the telescope in our program input language:

```
initp    mult
  0 0 0 0 0 0
stm      mult
  2.50000000000000     -1.00000000000000E-02   1.00000000000000
  0.200000000000000     1.00000000000000      -2.50000000000000
qm1      mult
  1.00000000000000     -0.374356314677254      2.00000000000000
  0.100000000000000     1.00000000000000       0.500000000000000
qm2      mult
  1.00000000000000      0.261552436692153      2.00000000000000
  0.100000000000000     1.00000000000000       3.00000000000000
int      int
 8 5   1 0 0  200 0
#lines
point
   1*initp      1*stm       1*qm1       1*qm2       1*int
```

Fig.1 shows the effect of the uncorrected second and third order aberrations on the beam divergence.



CONCLUSION

The significance of all this is that for the first time we can compute high order maps of realistic multipole elements with overlapping fringe fields. Since the routines have been written for arbitrary m, a basis has been laid for realistic calculations beyond fifth order. Since the reference trajectory, and the map about it are calculated by numerical integration through the general magnetic field, we can now calculate the curved reference trajectories in a general dipole magnet, as well as the offset reference trajectories needed for misalignment tolerance studies.

REFERENCES

1. A. J. Dragt, "Lectures on Nonlinear Orbit Dynamics," 1981 Fermilab Summer School on High Energy Particle Accelerators, AIP Conference Proceedings, vol. 87 (1982).
2. A. J. Dragt, F. Neri, G. Rangarajan, D. R. Douglas, L. M. Healy, and R. D. Ryne, "Lie Algebraic Treatment of Linear and Nonlinear Beam Dynamics," Ann. Rev. Nuc. Part. Sci. 38, 455 (1988).
3. R. D. Ryne, "Lie Algebraic Treatment of Space Charge," Ph.D. Thesis, University of Maryland (1987).
4. P. Walstrom, "Magnetic Fields and Inductances of Cylindrical Current Sheet Magnets", to be published as a Los Alamos National Laboratory Unclassified Report (1990).